Polarons in Φ_2^4

1989 J. Phys. A: Math. Gen. 22 L1153

(http://iopscience.iop.org/0305-4470/22/24/004)

J. Phys. A: Math. Gen. **22** (1989) L1153–L1157. Printed in the UK

LETTER TO THE EDITOR

Polarons in Φ_2^4

C Aragão de Carvalho†‖, C A Bonato‡¶ and G B Costamilan§

† Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544, USA
‡ Department of Mathematics, University of California at Irvine, Irvine, CA 92717, USA
§ Centro Brasileiro de Pesquisas Físicas, Rua Dr Xavier Sigaud 150, 22290, Rio de Janeiro, RJ, Brazil

Received 12 September 1989

Abstract. We show that broken φ_2^4 in a constant external field has the same polaron solutions as when coupled supersymmetrically to fermions. We use this to derive a mean-field scheme that yields the exact solution to the classical equations of the fermion-boson system. We explain the occurrence of similar solutions in the two theories by means of inverse-scattering arguments and comment on applications to polymer models.

Polarons (bounces) are non-topological solitons. They can be viewed as kink-antikink pairs emerging as exact solutions of nonlinear equations and can be found in two-dimensional models used in particle [1-4] and condensed matter [5-8] physics to describe the interaction of fermions and bosons. Although numerical calculations suggest that they exist for a variety of values of the couplings involved, analytic solutions have been obtained for a special relation between Yukawa and scalar couplings. In [8] it was shown that, if that relation holds, an infinite number of scalar Lagrangian densities, Yukawa coupled to fermions, will admit them as solutions to their coupled equations of motion. It turns out that, for the simple choice of a broken φ_2^4 Yukawa coupled to (Majorana) fermions, this relation corresponds to the supersymmetry condition and yields the Wess–Zumino [9] model. As a result, polarons are extrema for the Wess–Zumino model in its simplest version and should be considered in semiclassical computations.

However, they can appear in a much simpler situation: as exact solutions to the equation of motion of broken φ_2^4 coupled to a *constant* external current [10–12]. Indeed,

$$\ddot{\varphi}-\varphi''+\lambda\varphi(\varphi^2-\varphi_0^2)=-j \qquad (1)$$

has a *time-independent* solution. Consider the first integral of (1) for a static field

$$\frac{1}{2}\varphi'^2-\frac{\lambda}{4}(\varphi^2-\varphi_0^2)^2-j\varphi=E. \qquad (2)$$

This is just the expression of energy conservation for a point particle at position φ, evolving in a 'time' given by the spatial coordinate, x. The potential, in our mechanical analogue, is minus the scalar potential

$$V(\varphi)=\frac{\lambda}{4}(\varphi^2-\varphi_0^2)^2+j\varphi \qquad (3)$$

‖ On leave from Departamento de Física, Pontifícia Universidade Católica, Caixa Postal 38071, 22453, Rio de Janeiro, RJ, Brazil.
¶ On leave from Departamento de Física, Universidade Federal da Paraíba, 58000, João Pessoa, PB, Brazil.

0305-4470/89/241153+05$02.50

i.e. a double well with asymmetric minima; turned upside down, it has the shape of the Sugar Loaf. Denoting global and local minima by φ_1 and φ_2, respectively, we shall be interested in a solution satisfying $\varphi = \varphi_2$ and $\varphi' = 0$ at $x = \pm\infty$. This allows us to compute E. Furthermore, since φ_2 is a minimum, we obtain j as a function of φ_2. Equation (2) can, then, be written in terms of φ_0, φ_2 and λ and integrated. After some algebra, we obtain

$$\varphi = \varphi_2 - \varphi_p[\tanh(\xi + \xi_0) - \tanh(\xi - \xi_0)] \tag{4}$$

where $\varphi_{\mathrm{p}} = \sqrt{(3\varphi_2^2 - \varphi_0^2)/2}$ and also

$$\xi = \sqrt{\frac{\lambda}{2}}\,\varphi_p(x - X) \qquad \xi_0 = \tfrac{1}{2}\cosh^{-1}\left[\left(\frac{2}{(\varphi_0^2/\varphi_2^2) - 1}\right)^{1/2}\right]. \tag{5}$$

The integration constant, X, is arbitrary and defines the position of the centre of the polaron. The quantity $x_0 = \xi_0(\sqrt{\lambda/2}\varphi_{\mathrm{p}})$ is half the distance between the centres of the kink and antikink that make up the polaron. Equation (4) has the same functional form as the polaron solutions mentioned in the introduction.

We can now use this last observation to deal with the equations of motion of broken φ_2^4 (the σ-model), Yukawa coupled to fermions. The Lagrangian density

$$\mathscr{L} = \tfrac{1}{2}\partial_\mu\varphi^\mu\varphi - \frac{\lambda}{4}(\varphi^2 - \varphi_0^2)^2 + \bar{\psi}(\mathrm{i}\gamma^\mu\partial_\mu - g\varphi)\psi \tag{6}$$

leads to the coupled system

$$\partial_\mu\partial^\mu\varphi + \lambda\varphi(\varphi^2 - \varphi_0^2) = -g\bar{\psi}\psi \tag{7}$$

$$(\mathrm{i}\gamma^\mu\partial_\mu - g\varphi)\psi = 0. \tag{8}$$

We can try a mean-field-type approach by replacing the LHS of (7) with a constant, $(-j)$. We may, then, use the results of the previous paragraph and solve for φ. This can now be inserted in (8). The result (equation (4)) can now be inserted in (8). Since our scalar field is static, we obtain $H_D\zeta = -\omega\zeta$, $\psi(t, x) = \mathrm{e}^{-\mathrm{i}\omega t}\zeta(x)$ and

$$H_D = \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix} \tag{9}$$

where we have used the Weyl representation for the Dirac matrices. If we square the Hamiltonian, we obtain two Schrödinger equations which are related to each other. The problem is effectively one dimensional and can be viewed as an example of supersymmetric quantum mechanics: the operators $DD^\dagger$ and $D^\dagger D$ have the same spectra except, possibly, for eventual zero eigenmodes. They are of the form

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + g^2(\varphi_2^2 - 2\varphi_{\mathrm{p}}^2) + C_\pm^{(+)}\tanh^2(\xi_+) + C_\pm^{(-)}\tanh^2(\xi_-)\right]u_\pm = \omega^2 u_\pm \tag{10}$$

with $\xi_\pm = \xi \pm \xi_0$ and the remaining $\pm$ subscripts denoting each Schrödinger equation. The constants $C_\pm^{(\pm)}$ (subscript denotes equation) obey

$$C_+^{(\pm)} = C_-^{(\mp)} = (g \mp \sqrt{\lambda/2})g\varphi_{\mathrm{p}}^2. \tag{11}$$

If we now impose the relation $\lambda = 2g^2$, there will be a decoupling of the variables $\xi_\pm$ and we end up with two independent Pöschl–Teller potentials. They are exactly solvable [13], with bound states given by

$$u_\pm = N_\pm\,\mathrm{sech}(\xi_\mp) \tag{12}$$

where $N_\pm$ are normalisation constants. The corresponding eigenenergies are

$$\omega^2 = \frac{g}{2}(\varphi_0^2 - \varphi_2^2). \tag{13}$$

From the solutions of two Schrödinger problems we may reconstruct the bound states of the Dirac equation by using the Hamiltonian (9)

$$\zeta_+ = \left(u_-, -\frac{1}{\omega} Du_+\right) \qquad \zeta_- = \left(-\frac{1}{\omega} D^\dagger u_-, u_-\right) \tag{14}$$

where ω stands for $\sqrt{\omega^2}$. Thus, there exist two charge-conjugate bound states, with energies $\pm\omega$, whose wavefunctions are respectively given by

$$\zeta_\pm = \mathcal{N}\begin{pmatrix} \operatorname{sech}(\xi_-) \\ \pm\operatorname{sech}(\xi_+) \end{pmatrix} \tag{15}$$

with $\mathcal{N}\mathcal{N}^* = \frac{1}{4}\sqrt{(\lambda/2)}\varphi_p$ to account for normalisation. We may now feed those states back into the RHS of (7) and proceed to a second iteration of our method. Although the RHS is no longer a constant, it is closely related to our initial ansatz for φ

$$\bar{\psi}_+\psi_+ = 2\mathcal{N}\mathcal{N}^* \operatorname{sech}(\xi_+)\operatorname{sech}(\xi_-) = -\frac{g}{2\sinh(2\xi_0)}(\varphi - \varphi_2). \tag{16}$$

Using this, we may rewrite (7) in the form of (1) by changing the values of mass and current. Equation (7) will then have a solution of the same form as before, but with a different set of parameters. All one has to do is to replace φ_2, φ_p and ξ_0 by new values $\tilde{\varphi}_2$, $\tilde{\varphi}_p$ and $\tilde{\xi}_0$, keeping the same value for g (and $\lambda = 2g^2$), in (4). Substituting into (7), one obtains

$$\tilde{\varphi}_2^3 - \tilde{\varphi}_2\varphi_0^2 = 0 \tag{17}$$

$$3\tilde{\varphi}_2^2 - 2\tilde{\varphi}_p^2 - \varphi_0^2 - \frac{1}{4\sinh(2\tilde{\xi}_0)} = 0 \tag{18}$$

$$\tilde{\varphi}_2 - \frac{\tilde{\varphi}_p}{\tanh(2\tilde{\xi}_0)} = 0. \tag{19}$$

From those, we obtain

$$\tilde{\varphi}_2^2 = \varphi_0^2 \qquad \tilde{\varphi}_p^2 = \tfrac{1}{8}(4\varphi_0^2 \pm \sqrt{16\varphi_0^4 - 1}) \qquad \cosh(2\tilde{\xi}_0) = 8\tilde{\varphi}_p^2. \tag{20}$$

There are two possible solutions, corresponding to fermion bound states of energies satisfying $\omega^2 = g^2\tilde{\varphi}_p^2$. They are the same as the ones found in [2] by inverse-scattering methods. There, it was shown that only the solution with the plus sign in (20) is stable. The other decays into a kink-antikink pair. The fermion bound states come out of the solution of the Dirac equation, which can be obtained as before. Our iterative procedure converges in its second iteration, as a consequence of the relation $\lambda = 2g^2$.

Apart from assuring the convergence of our procedure, the relation between the couplings is equivalent to the supersymmetry condition for the fermion-boson model. Indeed, the underlying quantum mechanical supersymmetry, which relates the spectra of our two Schrödinger operators, leads to a self-conjugate spectrum for the Dirac equation. That allows us to work with Majorana fermions (note that only the positive energy state was used in (7)). Together with the condition on the couplings, this results in a Lagrangian density which can be rewritten in the form

$$\mathscr{L}_{WZ} = \tfrac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + U(\varphi) + \bar{\psi}[\mathrm{i}\gamma^\mu\partial_\mu - U'(\varphi)]\psi \tag{21}$$

where $U(\varphi)=(g^2/2)(\varphi^2-\varphi_0^2)^2$. One immediately recognises the Wess-Zumino Lagrangian in two dimensions.

It remains to explain why a constant external current induces the same form of solution as the richer model with fermion interactions. This can be best understood in the language of inverse scattering. It suffices to know that the Lagrangian leading to (1) can be completely written in terms of scattering data for an auxiliary Dirac problem, whose Yukawa potential is the scalar field. Indeed,

$$\mathscr{L}(j)=\tfrac{1}{2}\partial_\mu\varphi\partial^\mu\varphi+[V(\varphi)-V(\varphi_2)] \tag{22}$$

with V defined in (3), yields a Lagrangian that can be written as

$$L(j)=L_3+2(\varphi_2^2+\varphi_0^2)L_1+jL_0 \tag{23}$$

where the L_i can then be expressed as [8]

$$L_0=\int_{-\infty}^{+\infty}\mathrm{d}x(\varphi-\varphi_2)=\frac{1}{2g}\ln\left(\frac{T_+(-\mathrm{i}g\varphi_2)}{T_-(+\mathrm{i}g\varphi_2)}\right) \tag{24}$$

$$L_1=\int_{-\infty}^{+\infty}\mathrm{d}x(\varphi^2-\varphi_2^2)=-\frac{1}{g^2}\left\{\int_{-\infty}^{\infty}\frac{\mathrm{d}q}{2\pi}P_+(q)+2K_++[(+)\leftrightarrow(-)]\right\} \tag{25}$$

$$\begin{aligned}L_3&=\int_{-\infty}^{+\infty}\mathrm{d}x\left[\left(\frac{d\varphi}{dx}\right)+g^2(\varphi^2-\varphi_2^2)^2\right]\\&=-\frac{1}{g^2}\left\{\int_{-\infty}^{\infty}\frac{\mathrm{d}q}{\pi}q^2P_+(q)-\tfrac{4}{3}(K_+)^3+[(+)\leftrightarrow(-)]\right\}\end{aligned} \tag{26}$$

with $(K_\pm)^2+g^2\varphi_2^2=\omega^2$. The expression for $P_\pm(q)=\ln(|T_\pm(q)|^2)$ depends on the transmission coefficients, $T_\pm(q)$, associated with the Dirac equation. They admit an integral representation [14]

$$\ln T_\pm(k)=\int_{-\infty}^{\infty}\frac{\mathrm{d}q}{2\pi\mathrm{i}}\frac{P_\pm(q)}{q-k}+\ln\left(\frac{k+\mathrm{i}K_\pm}{k-\mathrm{i}K_\pm}\right). \tag{27}$$

Note that we subtract an infinite constant from the Lagrangian so as to set it to zero at the boundary value φ_2. Extremising the action with respect to scattering variables does yield the condition that the potential of the Dirac equation should be reflectionless. This is equivalent to the relation $\lambda=2g^2$ and guarantees the existence of polaron solutions of the form shown in (4) (see [1-4, 7, 8]). It is the same strategy used in proving the existence of polarons in fermion-boson models, where the fermionic contribution can also be expressed in terms of scattering data and is also extremised by reflectionless potentials. It should, thus, be no surprise that the two systems have solutions of the same form.

The results we have described find an interesting application in the physics of polymers. It was shown in [8] that the models of [5] and [6] both had polaron solutions because the valence π-electrons of the former, when integrated out, generated an effective phonon potential with the characteristics of that of the latter. Thus, to derive polaron solutions, we could do away with valence electrons, modify the phonon potential and only keep the mid-gap (bound) states. We now see that a further reduction is possible, which replaces those remaining fermion states with a constant (mean-field) external current and still preserves the polarons. In fact, this observation was recently used [15] to construct approximate solutions to coupled QCD equations that bear a striking resemblance to those of polymer models. Furthermore, the role of the relation between couplings, which allowed for the construction of explicit solutions, presents us with an example of an effective supersymmetric model that describes fermion–boson dynamics and possesses classical solutions which, to our knowledge, have not been explored before.

This work is partially supported by CNPq and FINEP.

References

[1] Dashen R F, Hasslacher B and Neveu A 1975 *Phys. Rev.* D **12** 2443
[2] Campbell D K and Liao Y-T 1976 *Phys. Rev.* D **14** 2093
[3] Shei S S 1976 *Phys. Rev.* D **14** 535
[4] Ayoama S 1978 *Prog. Theor. Phys.* **60** 523
[5] Su W, Schrieffer R J and Heeger A 1979 *Phys. Rev. Lett.* **42** 1698
[6] Krive I V and Rozhavskii A S 1980 *Zh. Eksp. Teor. Fiz. Pis. Red.* **31** 647 (1980 *JETP Lett.* **31** 610)
[7] Campbell D K and Bishop A R 1982 *Nucl. Phys.* B **200**[FS4] 297
[8] Aragão de Carvalho C 1989 *Nucl. Phys.* B **324** 729
[9] Wess J and Zumino B 1974 *Phys. Lett.* **49B** 52
[10] Bonato C A, Thomaz M T and Malbouisson A P C 1989 The bounce and its negative eigenvalue: a new approach *Phys. Rev.* D submitted
[11] Costamilan G B 1989 Sólitons e Polarons em Polímeros *MSc Thesis* CBPF (unpublished)
[12] Aragão de Carvalho C 1988 *Acta Phys. Polon.* B **19** 875
[13] Morse P M and Feshbach H 1953 *Methods of Theoretical Physics* (New York: McGraw-Hill)
[14] Frolov I S 1972 *Dokl. Akad. Nauk. SSSR* **207** 44 (1972 *Sov. Math. Dokl.* **13** 1468)
[15] Aragão de Carvalho C 1989 Instantons and their Molecules in QCD *Princeton University preprint* PUPT 1142